

WEIGHTED PLURIPOTENTIAL THEORY ON COMPLEX KÄHLER MANIFOLDS

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ABSTRACT. We introduce a weighted version of the pluripotential theory on compact Kähler manifolds developed by Guedj and Zeriahi. We give the appropriate definition of a weighted pluricomplex Green function, its basic properties and consider its behaviour under holomorphic maps. We also establish a generalization of Siciak's H-principle.

INTRODUCTION

Recently there has been significant progress in weighted pluripotential theory on \mathbb{C}^N which was originally developed in [Si1],[Si2] and generalized to parabolic manifolds in [Ze]. Specifically, we refer to [BL], [B11], [B12], [Bra], [MS]. Concurrently, pluripotential theory on a compact Kähler manifold X based on quasisubharmonic functions has been explored in [GZ1], [GZ2],[Ko] and [HKH](see also applications in [Be1], [Be2], [BB]). The goal of our article is to develop a framework which would allow for a unified treatment of both generalizations of the classical theory and would also allow one to create an analog of the psh-homogeneous pluripotential theory. We will start by showing that a weighted pluripotential theory on \mathbb{C}^N extends naturally to a pluripotential theory on \mathbb{CP}^N with a suitably modified weight. In turn this extends to a homogeneous pluripotential theory in the universal line bundle over \mathbb{CP}^N , whose charts are biholomorphic to \mathbb{C}^{N+1} . We will generalize these results to projective algebraic manifolds.

We define a weighted pluricomplex Green function on a compact complex manifold X with a Kähler form ω . The definition is formulated in terms of a mild function (see Definition 1). However, many results of our theory hold without requiring that Q be mild. For a mild function Q and a Borel set $K \subset X$ the weighted pluricomplex Green function is

$$V_{K,\omega,Q} = \sup\{\phi \in PSH(X, \omega) : \phi \leq Q \text{ on } K\}.$$

Basic properties of $V_{K,\omega,Q}$ are stated and proved in Section 1, followed by the extension of the weighted pluripotential theory in \mathbb{C}^N to a suitable weighted pluripotential theory on \mathbb{CP}^N . We obtain more specific results, in particular a generalized Siciak's H-principle and some approximation results, in the case when X admits a positive line bundle (which by Kodaira's imbedding theorem is equivalent to X being projective algebraic).

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Preliminary version.

The initial motivation for our work was the similarity between Theorem 2.12 in [Bra] and Theorem 1 in [St1]; both of which are generalized versions of Theorem 5.3.1 in [Kl]. We succeeded in proving the following result (Theorem 5, Section 2) which gives the above mentioned theorems as special cases.

Theorem: Let (X, ω) be a compact complex Kähler manifold and $f : X \rightarrow X$ a holomorphic surjection. Assume that there exist α and β , $1 < \alpha \leq \beta$, such that $\alpha f_*(PSH(X, \omega)) \subset PSH(X, \omega)$ and $f^*(PSH(X, \omega)) \subset \beta \cdot PSH(X, \omega)$. Then for every Borel set $K \subset X$ and every mild function Q on X ,

$$\alpha V_{f^{-1}(K), \omega, f^*Q/\alpha}(x) \leq V_{K, \omega, Q} \circ f(x) \leq \beta V_{f^{-1}(K), \omega, f^*Q/\beta}.$$

1. WEIGHTED PLURICOMPLEX GREEN FUNCTIONS

Throughout the paper we assume that X is a connected compact complex Kähler manifold. Therefore we have on X (cf. [GF], VI.3) the fundamental form ω of a hermitian metric Γ on X with $\omega = i \sum_{j,k} \gamma_{jk} dz_j \wedge d\bar{z}_k$, satisfying $d\omega = 0$. It follows that in each coordinate neighborhood in X we can define a \mathcal{C}^∞ real-valued function ϕ such that $i\partial\bar{\partial}\phi = (1/2)dd^c\phi = \omega$. The functions ϕ are called local potentials of the Kähler metric Γ . Existence of smooth local potentials is in fact equivalent to Γ being Kähler: if the fundamental form ω of Γ satisfies $\omega = i\partial\bar{\partial}\phi$, then $d\omega = 0$. For example, the Fubini-Study metric on \mathbb{CP}^N is Kähler, since it has local potentials given by $\phi_j = \log(1 + \sum_{k \neq j} |z_{j,k}|^2)$ in the coordinate neighborhoods $U_j = \{Z_j \neq 0\}$ with $j = 0, 1, \dots, N$. Here $[Z_0 : \dots : Z_N]$ are homogeneous coordinates in \mathbb{CP}^N and $z_{j,k} := Z_k/Z_j$ in U_j . The set U_0 is identified with \mathbb{C}^N and $z_{0,k} =: z_k$, $k = 1, \dots, N$, are affine coordinates. We have $\phi_0 = \log(1 + \|z\|^2)$ for $z \in \mathbb{C}^N$. Let ω be a closed real $(1, 1)$ current on X with continuous local potentials. From [GZ1], the class of ω -plurisubharmonic functions is defined as

$$PSH(X, \omega) = \{v \in L^1(X, \mathbb{R} \cup \{-\infty\}) : dd^c v \geq -\omega \text{ and } v \text{ is upper semicontinuous}\}.$$

The ω -pluricomplex Green function of a Borel set $K \subset X$ is defined as

$$V_{K, \omega}(x) = \sup\{v(x) : v \in PSH(X, \omega) : v|_K \leq 0\}$$

Consider the class of $PSH(X, \omega)$, where ω is a Kähler form on X with local potentials $\phi_j : U_j \rightarrow \mathbb{R}$, where $\{U_j\}_{j=0}^m$ is an open cover of X by coordinate neighborhoods.

Definition 1. Let $Q : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $\exp(-Q + \phi_j)$ is continuous in U_j , $j = 1, \dots, m$ and $\{Q \neq +\infty\}$ is not a pluripolar subset of X . We will call Q satisfying these assumptions a mild function.

Mild functions are necessarily lower semicontinuous.

Definition 2. For a mild function Q on X and for a Borel set $K \subset X$ let us define the weighted ω -pluricomplex Green function as

$$V_{K, \omega, Q} = \sup\{\phi \in PSH(X, \omega) : \phi \leq Q \text{ on } K\}.$$

The following properties are direct consequences of our definition of $V_{K, \omega, Q}$.

Proposition 1. Let K, K_1, K_2 be Borel subsets of X and Q, Q_1, Q_2 be mild functions.

- i) If $Q_1 \leq Q_2$ on K then $V_{K, \omega, Q_1} \leq V_{K, \omega, Q_2}$.

- ii) If $K_1 \subset K_2$ then $V_{K_2, \omega, Q} \leq V_{K_1, \omega, Q}$.
- iii) Let Q be a mild function that belongs to the class $PSH(X, \omega)$. Then $V_{X, \omega, Q} = Q$.
- (iv) Let ω' be cohomologous to ω , $\omega' = \omega + dd^c \xi$ for $\xi \in L^1(X)$. If ξ is mild and continuous, then $V_{K, \omega', Q} = V_{K, \omega, Q - \xi} + \xi$.

We continue to establish basic properties of the weighted pluricomplex Green function in Propositions 2 and 3.

Proposition 2. *Let K be a Borel set in X and Q a mild function on X . If K is not $PSH(X, \omega)$ -polar then $V_{K, \omega, Q}^* \in PSH(X, \omega)$.*

Proof. By Choquet's lemma there exists an increasing sequence of functions $\phi_j \in PSH(X, \omega)$ such that $\phi_j \leq Q$ on K and

$$V_{K, \omega, Q}^* = (\lim_{j \rightarrow \infty} \phi_j)^*.$$

It follows from Proposition 2.6(2) in [GZ1] that $V_{K, \omega, Q}^* \in PSH(X, \omega)$. \square

Proposition 3. *Let E be a Borel subset of X and P a $PSH(X, \omega)$ polar set. Then we have*

$$V_{E \cup P, \omega, Q}^* = V_{E, \omega, Q}^*.$$

Proof. Recall that a set P is said to be $PSH(X, \omega)$ -polar if it is included in the $-\infty$ -locus of some function $\psi \in PSH(X, \omega)$ which is not identically $-\infty$ on X . By Prop 1(ii) we have $V_{E \cup P, \omega, Q}^* \leq V_{E, \omega, Q}^*$. Consequently, we need to establish $V_{E, \omega, Q}^* \leq V_{E \cup P, \omega, Q}^*$. Suppose $u \in PSH(X, \omega)$ with $u \leq Q$ on E and let $v \in PSH(X, \omega)$ such that $P \subset \{v = -\infty\}$. We may assume $v \leq Q$ on E . Then for each $\epsilon > 0$,

$$(1 - \epsilon)u + \epsilon v \leq V_{E \cup P, \omega, Q} \leq V_{E \cup P, \omega, Q}^*.$$

Therefore $u \leq V_{E \cup P, \omega, Q}^*$ on X and by taking the supremum, $V_{E, \omega, Q}^* \leq V_{E \cup P, \omega, Q}^*$. \square

Let us now show how weighted pluripotential theory on \mathbb{C}^N can be extended to a suitable weighted pluripotential theory on \mathbb{CP}^N . Recall that in the weighted theory on \mathbb{C}^N one begins with an admissible weight function on a closed set $K \subset \mathbb{C}^N$. An admissible weight w is a nonnegative upper semicontinuous function w on \mathbb{C}^N with $\{z \in K : w(z) > 0\}$ non-pluripolar and satisfying the boundedness condition $\lim_{|z| \rightarrow \infty} |z|w(z) = 0$ if K is an unbounded set (cf. [BL], [B11], [ST]). The weighted pluricomplex Green function of K is defined as

$$V_{K, Q} = \sup\{u \in \mathcal{L}, u \leq Q \text{ on } K\}.$$

where $Q = -\log w$.

In the homogeneous coordinates $[Z_0 : \dots : Z_N]$ on \mathbb{CP}^N (with the usual identification $\mathbb{C}^N \simeq \{Z_0 \neq 0\}$ and affine coordinates $z_j = Z_j/Z_0$, $j = 1, \dots, N$) let $\tilde{w}(Z_0 : \dots : Z_N) = w(z_1, \dots, z_N)/|Z_0|$ in $\{Z_0 \neq 0\}$, where w is nonnegative and upper semicontinuous with $\{w > 0\}$ non-pluripolar, but not necessarily satisfying the boundedness condition. The expression $W(Z) = \|Z\| \tilde{w}(Z)$ defines a homogeneous function of order 0 in $\mathbb{C}^{N+1} \setminus \{Z_0 = 0\}$. We have $W(Z) = \varphi_0(z) + \log w(z)$ for $Z_0 \neq 0$, where $\varphi_0(z) = (1/2) \log(1 + |z|^2)$. We take

$$\sqrt{|Z_1|^2 + \dots + |Z_N|^2} \tilde{w}(0 : Z_1 : \dots : Z_N) = \limsup_{0 \neq Y_0 \rightarrow 0, Y_j \rightarrow Z_j} \|Y\| \tilde{w}(Y), Y = (Y_0, \dots, Y_N)$$

to obtain an upper semicontinuous function (still denoted by W) globally on \mathbb{CP}^N , with all values greater or equal to 0. The boundedness condition is equivalent to the property that this global function is identically zero on the hyperplane $\{Z_0 = 0\}$. This is because $\lim_{|z| \rightarrow \infty} |z|w(z) = \lim_{|z| \rightarrow \infty} \sqrt{1 + |z|^2}w(z)$. We will assume a weaker condition, namely that W is bounded in \mathbb{CP}^N .

The following example demonstrates that the boundedness condition is too restrictive when constructing a weighted pluripotential theory on complex manifolds.

Example 1. Let ω_{FS} be the Fubini-Study Kähler form on $X = \mathbb{CP}^N$ with local potentials ϕ_j as above and let K be a subset of $\mathbb{C}^N \subset \mathbb{CP}^N$. For $Z \in \mathbb{CP}^N$ define $Q_j(Z) = \phi_j(Z)$, $j = 0, \dots, N$, so that $Q_0(z) = (1/2) \log(\sqrt{1 + \|z\|^2})$ for $z \in \mathbb{C}^N$. The natural 1-to-1 correspondence between $PSH(X, \omega_{FS})$ and the class $\mathcal{L}(\mathbb{C}^N)$ of plurisubharmonic functions with logarithmic growth at infinity, presented explicitly in Example 1.2 in [GZ1], gives the following:

$$\begin{aligned} V_{K, Q_0}(x) &= \sup\{u(x) : u \in \mathcal{L}(\mathbb{C}^N), u(z) \leq \log \sqrt{1 + \|z\|^2} \quad \forall z \in K\} \\ &= \sup\{u(x) : u \in \mathcal{L}(\mathbb{C}^N), u(z) - (1/2) \log(1 + |z|^2) \leq 0 \quad \forall z \in K\} \\ &= \sup\{v(x) + (1/2) \log(1 + |x|^2), v \in PSH(\mathbb{CP}^N, \omega_{FS}) : v|_K \leq 0\} \\ &= V_{K, \omega_{FS}}(x) + (1/2) \log(1 + |x|^2) \end{aligned}$$

for every $x \in \mathbb{C}^N$. Assume now that K is not $PSH(\mathbb{CP}^N, \omega_{FS})$ -polar. Then $V_{K, \omega_{FS}}^* \in PSH(\mathbb{CP}^N, \omega_{FS})$ and $V_{K, Q_0}^* \in \mathcal{L}(\mathbb{C}^N)$. For a point Z on the hyperplane at infinity $\{Z_0 = 0\}$ we get

$$V_{K, \omega_{FS}}^*(Z) = \limsup_{x \rightarrow Z, x \in \mathbb{C}^N} (V_{K, Q_0}^*(x) - (1/2) \log(1 + |x|^2)).$$

Note that the function $w(z) = \exp(-Q_0(z))$ in our example does not satisfy the boundedness condition in \mathbb{C}^N . Indeed, the function $\|Z\|w(z) = \exp(-Q_j(Z) + \phi_j(Z))$ for $Z \in U_j$, $j = 0, \dots, N$ is a constant function 1 on \mathbb{CP}^N (which of course is continuous, but never 0). We draw the reader's attention to the paper [Bl2], in which a relation between weighted theory in \mathbb{C}^N and standard pluripotential theory in \mathbb{C}^{N+1} is outlined. Examples considered in the Section 5 of that paper deal with a weight function w which is given as the Hartogs radius of a domain with balanced fibers in \mathbb{C}^{N+1} (for the definition and basic properties, see [Sh]). Such a function is upper semicontinuous, but as shown in [Bl2], does not have to satisfy the boundedness condition on \mathbb{C}^N . Furthermore, the results of [Si2] as well as [MS] were obtained without assuming the boundedness condition. It thus seems reasonable to weaken this condition when working on complex manifolds. In [Gu] a notion of a 'convex' hull with respect to a closed real $(1, 1)$ -current T is considered where the functions f defining the hull satisfy the condition that $\exp(f + \phi)$ are continuous, with ϕ continuous local potentials for T . We adopted an analogous condition as a part of our definition of a mild function.

The method demonstrated in Example 1 can also be used to prove the following:

Proposition 4. Let $K \subset \mathbb{C}^N \cong \{Z_0 \neq 0\}$. For a mild function Q on \mathbb{CP}^N with respect to $\omega = \omega_{FS}$ define

$$q(z_1, \dots, z_N) = q(Z_1/Z_0, \dots, Z_N/Z_0) = Q(Z) - \log(\|Z\|/|Z_0|), \quad Z_0 \neq 0.$$

Conversely, for a lower semicontinuous q on \mathbb{C}^N , consider

$$Q(Z) = q(Z_1/Z_0, \dots, Z_N/Z_0) + \log \|Z\| + \log |Z_0|,$$

together with its lower semicontinuous regularization as $Z_0 \rightarrow 0$. Then $V_{K,q}(x) = V_{K,\omega,Q}(x) = (1/2) \log(1 + \|x\|^2)$, $x \in \mathbb{C}^N$.

Consider now a holomorphic line bundle L over a compact Kähler manifold X . Recall that a (singular) metric on L can be given (cf. [De], [DPS]) by a collection of real-valued functions $h = \{h_j\}$ on X , defined in a trivializing cover $\{U_j\}$, such that $h_j = h_i + \log |g_{ij}|$, where g_{ij} are transition functions for L . The metric is called positive if all h_j are plurisubharmonic. (The notion of positivity is used here in the weak sense.) In particular, a smooth metric $\{\phi_j\}$ such that $\omega = dd^c \phi_j$ is a Kähler form will be positive.

If L is a positive line bundle and $\omega = [c_1(L)]$, there is a 1-to-1 correspondence between the family of all positive metrics on L and the class $PSH(X, \omega)$. In the case of $X = \mathbb{CP}^N$ with the Fubini-Study form ω , this correspondence is equivalent to the H -principle due to Siciak ([Si3]).

Proposition 5. (cf. [G], property (iv) pg 456): *Let h be a logarithmically homogeneous plurisubharmonic nonnegative function on \mathbb{C}^{N+1} . Then h defines a positive metric on \mathbb{CP}^N . Conversely, each positive metric on \mathbb{CP}^N defines a logarithmically homogeneous psh function on \mathbb{C}^{N+1} .*

Proof. By logarithmic homogeneity we have,

$$v(Z_0/Z_k, \dots, 1, \dots, Z_N/Z_k) = v(Z) - \log |Z_k| \text{ in } \{Z_k \neq 0\}$$

Hence $v_k = v_i + \log |Z_k/Z_i|$ in $U_i \cap U_k$ and all v_i are plurisubharmonic. To prove the converse, take $h_0 = h|_{U_0}$. The function $v(Z) = h_0(Z) + \log |Z_0|$ in U_0 , and $v(0, Z_1, \dots, Z_N) = \limsup_{(Y_0 \rightarrow 0, Y_j \rightarrow Z_j)} v(Y_0, Y_1, \dots, Y_N)$ is plurisubharmonic. Since it also satisfies $v(\lambda Z) = v(Z) + \log |\lambda|$ for $\lambda \in \mathbb{C}$ our proof is complete. \square

By Example 1.2 in [GZ1], the class $\mathcal{L}(\mathbb{C}^N)$ corresponds in a 1-to-1 manner with the class of $PSH(\mathbb{CP}^N, \omega)$ functions, which in turn correspond in a 1-to-1 manner with positive metrics on the (positive) hyperplane bundle over \mathbb{CP}^N . Thus Proposition 5 establishes a 1-to-1 correspondence between logarithmically homogeneous functions \tilde{v} on \mathbb{C}^{N+1} and functions v in the class $\mathcal{L}(\mathbb{C}^N)$ so that $\tilde{v}(1, z) = v(z)$ for $z \in \mathbb{C}^N$, that is, the H -principle.

If L is a positive line bundle over X , then its dual L' is negative ([GF], Prop. VI.6.1 and VI.6.2). Hence there exists a system of trivializations $\theta_i : L' |_{U_i} \rightarrow U_i \times \mathbb{C}$ with transition functions $G_{ik} = g_{ik}^{-1} = g_{ki}$ and a smooth metric $\{h_i\}$ on L such that the smooth function $\chi_h : L' \rightarrow \mathbb{R}$, defined as $\chi_h \circ \theta_i^{-1}(x, t) = H_i(x) \cdot |t|^2$, is strictly plurisubharmonic outside the zero section of L' , where $H_i(x) = \exp 2h_i(x)$, $x \in U_i$. As a simple example of a negative line bundle we can take the universal line bundle over \mathbb{CP}^N , $\mathcal{O}(-1) := \{([Z], \xi) \in \mathbb{CP}^N \times \mathbb{C}^N : \xi \in \mathbb{C} \cdot Z, Z \in \mathbb{C}^{N+1} \setminus \{0\}, [Z] = \mathbb{C}^* \cdot Z\}$. That is, the fiber of $\mathcal{O}(-1)$ over a point $[Z] \in \mathbb{CP}^N$ is the complex line in \mathbb{C}^{N+1} generated by (Z_0, \dots, Z_N) . The function $\chi \circ \theta_i^{-1}(Z, t) = |t|^2 |Z_i|^{-2} \|Z\|^2$ for $Z_i \neq 0$, associated with the Fubini-Study metric on the dual line bundle $\mathcal{O}(1)$ over \mathbb{CP}^N , is plurisubharmonic.

Next we establish a generalization of Siciak's H -principle.

Theorem 1. (cf. [GF], Prop. VI.6.1): Let L be a positive line bundle over a compact Kähler manifold X and let $d > 0$. Let \mathcal{H}_d^+ denote the family of all functions $\chi \in PSH(L')$ which are nonnegative, not identically 0 and absolutely homogeneous of order d in each fiber. Then there is a one-to-one correspondence between \mathcal{H}_d^+ and the class of positive metrics on L .

Proof. Consider a system of trivializations $\theta_i : L' |_{U_i} \mapsto U_i \times \mathbb{C}$ with transition functions $G_{ik} = g_{ki} = 1/g_{ik}$. Let $\chi \in \mathcal{H}_d^+$. For $x \in U_i, t \neq 0$ define

$$H_i(x) := \chi \circ \theta_i^{-1}(x, t) / |t|^d.$$

Note that this expression does not depend on t . We have $\chi \circ \theta_i^{-1}(x, t) = \chi \circ \theta_k^{-1}(x, G_{ki}(x)t)$, hence by absolute homogeneity of order d , $H_k(x) = |G_{ki}(x)|^d H_i(x)$ in $U_i \cap U_k$. Taking $h_i = (1/d) \log H_i$ in U_i we get a collection of plurisubharmonic functions satisfying $h_k = \log |g_{ik}| + h_i$, i.e., a positive metric on L . Conversely, let $\{h_i\}$ be a metric on L . The function χ on L' defined as $\chi \circ \theta_i^{-1}(x, t) = \exp(dh_i(x)) \cdot |t|^d$ is plurisubharmonic if and only if h_i are, so for a positive metric the associated function χ is in \mathcal{H}_d^+ . \square

Unless otherwise indicated, we will work with $\mathcal{H}^+ := \mathcal{H}_1^+$. Note that if we take L' in Theorem 1 to be the universal line bundle \mathcal{U} over \mathbb{CP}^N , then the trivialization $\theta_i : \pi^{-1}(U_i) \mapsto U_i \times \mathbb{C}$ is given as $\theta_i(t(Z)) = ([Z_0 : \dots : Z_N], tZ_i)$. Hence for a function $\chi \in \mathcal{H}^+$ we have $\chi \circ \theta_i^{-1}([Z_0 : \dots : Z_N], t) = h_i([Z_0 : \dots : Z_N]) + \log |Z_i| + \log |t|$ for $Z_i \neq 0$, where h_i define a metric on \mathbb{CP}^N . By Proposition 5, over the chart $Z_0 \neq 0$ we get $\chi(tZ) = v(Z_1/Z_0, \dots, Z_N/Z_0) + \log |t|$ for $t \neq 0$ with v plurisubharmonic. That is, χ defines a logarithmically homogeneous psh function on \mathbb{C}^{N+1} .

For a positive holomorphic line bundle L over a compact Kähler manifold X there is a precise relation between the weighted pluricomplex Green function with respect to $\omega = [c_1(L)]$ of a Borel set K in X and a \mathcal{H}^+ -envelope of some associated set \tilde{K} in the dual bundle L' . It generalizes the formulas obtained by Bloom in ([Bl2]).

For the weight Q on X consider the collection $q_i = Q - \phi_i$, where $\omega = dd^c \phi_i$ in U_i and U_i form a trivializing cover for L . For $K \subset X$ define $\tilde{K} \subset L'$ by taking

$$\tilde{K} \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x, t) : x \in U_i \cap K, |t| = \exp(-q_i(x))\}.$$

This set is well defined, since $\theta_k^{-1}(x, t) = \theta_i^{-1}(x, G_{ki}(x)t)$. Hence if $x \in U_i \cap U_k \cap K$, then $|G_{ki}(x)t| = \exp(-q_i(x))$ if and only if $|t| = \exp(-q_k(x))$. Consider

$$H_{\tilde{K}} = \sup\{u \in PSH(L') : \exp u \in \mathcal{H}^+, u|_{\tilde{K}} \leq 0\}.$$

The following theorem gives the relationship between functions $H_{\tilde{K}}$ and $V_{K, \omega, Q}$.

Theorem 2. (cf. [Bl2], Thm 2.1): For all i ,

$$H_{\tilde{K}} \circ \theta_i^{-1}(x, t) = V_{K, \omega, Q}(x) + \log |t| + \phi_i(x)$$

Proof. By Theorem 1,

$$\begin{aligned} H_{\tilde{K}} &= \sup\{u : u \circ \theta_i^{-1}(x, t) = h_i(x) + \log |t|, \quad u|_{U_i \cap K} \leq 0\} \\ &= \sup\{u : u \circ \theta_i^{-1}(x, t) = h_i(x) + \log |t|, \quad h_i(x) \leq q_i, \forall i\} \end{aligned}$$

where h_i define a positive metric on L . Hence, for such h_i ,

$$\begin{aligned} H_{\tilde{K}} \circ \theta_i^{-1}(x, t) &= \sup\{h_i(x) : h_i(x) \mid_{K \cap U_i} \leq q_i\} + \log |t| \\ &= \sup\{v(x) + \phi_i(x) : v \in PSH(X, \omega), v \mid_K \leq Q\} + \log |t| \\ &= V_{K, \omega, Q}(x) + \log |t| + \phi_i(x), \quad \forall i. \end{aligned}$$

□

Theorem 2 allows us to study the behavior of the weighted pluricomplex Green functions as we vary the weight. Namely, we have the following:

Proposition 6. (cf. [Bl2], Cor 2.2): *Let $K \subset X$ be a Borel set. Suppose $\{Q_n\}, Q$ are mild functions with $Q_n \nearrow Q$. Then $\lim_{n \rightarrow \infty} V_{K, \omega, Q_n} = V_{K, \omega, Q}$.*

Proof. Consider the sets $K_n, M_n \subset L'$, where

$$M_n \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x, t) : x \in U_i \cap K, |t| \leq \exp(-q_i^{(n)}(x))\},$$

$$K_n \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x, t) : x \in U_i \cap K, |t| = \exp(-q_i^{(n)}(x))\}$$

where $q_i^{(n)} = Q_n - \phi_i, n \geq 0$. The sequence M_n is decreasing, with $\bigcap_{n=1}^{\infty} M_n = \tilde{M}_0$. By maximum principle (applied in each fiber), $H_{M_n} = H_{K_n}, n \geq 0$ (here we use the assumption of all Q_n being mild). For a function $u \in \mathcal{H}^+$ such that $u \leq 0$ on M_0 and an arbitrary $\varepsilon > 0$, there exists an n_0 such that for all $n \geq n_0$ we have $M_n \subset \{u < \varepsilon\}$. The function $u - \varepsilon$ is in \mathcal{H}^+ and for $n \geq n_0$ it satisfies $u - \varepsilon \leq H_{M_n} \leq \lim_{n \rightarrow \infty} H_{M_n} \leq H_{M_0}$, hence $\lim_{n \rightarrow \infty} H_{M_n} = H_{K_0}$. By Theorem 2, $\lim_{n \rightarrow \infty} V_{K, \omega, Q_n} = V_{K, \omega, Q_0}$. □

Proposition 7. (cf. [Bl2], Cor 2.4) *Let $Q_n, n \geq 0$ be mild functions on X such that $Q_n \searrow Q_0$. Then $V_{K, \omega, Q_0} = \lim_{n \rightarrow \infty} V_{K, \omega, Q_n}$.*

Proof. Since the potentials ϕ_i of ω are continuous, we have $H_{\tilde{K}}^* \circ \theta_i^{-1}(x, t) = V_{K, \omega, Q_0}^* + \log |t| + \phi_i$ for all i . We can assume that the set M_1 (see Proposition 6) is not ω -polar. By Proposition 2, $H_{K_0}^*$ is plurisubharmonic on L' . Let $H = \lim_{n \rightarrow \infty} H_{M_n}$. The function H is in \mathcal{H}^+ and satisfies $H \leq 0$ on $K_0 \setminus P$, where P is some pluripolar set. Hence $H \leq H_{K_0}^*$. □

Corollary 1. *Proposition 6, holds when the convergence $Q_n \searrow Q$, takes place quasi-everywhere on X , that is, outside some ω -polar set.*

Corollary 2. *Proposition 7 holds when the convergence $Q_n \nearrow Q$ takes place quasi-everywhere on X .*

2. APPROXIMATION AND PULLBACKS BY HOLOMORPHIC MAPS

In standard pluripotential theory in \mathbb{C}^N and its weighted generalization there is a function Φ_K such that $\log \Phi_K = V_{K, Q}$. The function Φ_K is given as the supremum of certain functions with 'regular' growth, that is, polynomials (when $Q \equiv 0$) or weighted polynomials (see Theorem 6.2 in [Si1], Theorem 2.8 in [Bl1], and Théorème 5.1 in [Ze]). In [GZ1] it is proven that $V_{K, \omega}(x) = \sup\{(1/n) \log \|s\|_{n\varphi}(x) : n \geq 1, s \in \Gamma(X, L^n), \sup_K \|s\|_{n\varphi} \leq 1\}$, where L is a positive holomorphic line bundle over a compact manifold X , $\omega = dd^c \varphi_j$ in a trivializing cover U_j is a (global) Kähler form and the norm $\|s\|_{n\varphi}$ of a section s of the tensor power L^n is

computed as follows: $\|s\|_{n\varphi} = |s_j| \exp(-n\varphi_j)$ in U_j . All such theorems are based on the possibility of approximation of general plurisubharmonic functions by so-called Hartogs functions, which are obtained by certain operations from functions of the type $\log |f|$ with f holomorphic (cf. [KL], theorem 5.1.6) Such approximation may be not always possible, but is possible for example in pseudoconvex domains in \mathbb{C}^N , as shown in [Bre]. Below, we will work in pseudoconvex neighborhoods of the zero section of L' to prove the following:

Theorem 3. *Let X, L, φ, ω be as above. Let Q be a mild function on X and let K be a compact subset of X . Then*

$$V_{K, \omega, Q} = \log \Phi_{K, \omega, Q} \text{ where } \Phi_K(x) = \sup_{n \geq 1} (\Phi_n(x))^{1/n}$$

with

$$\Phi_n(x) = \sup \{ \|s\|_{n\varphi}(x) : n \geq 1, s \in \Gamma(X, L^n), \sup_K \exp(-nQ) \|s\|_{n\varphi} \leq 1 \}.$$

Unlike [GZ1], in which the theorem was proved for $Q \equiv 0$, we will not use L^2 -estimates for the $\bar{\partial}$ -operator. Instead, we will apply the following lemma (cf. [Ze], Lemme 5.2, [Be1], Lemma 2.1 and 3.2):

Approximation Lemma. *Let X, ω, L be as above and let $v \in PSH(X, \omega) \cap C^\infty$ be such that $dd^c v + \omega$ is strictly positive. Then for every $\varepsilon > 0$ and every compact $K \subset X$ there exist N_1, \dots, N_m and s_1, \dots, s_m such that $s_j \in \Gamma(X, L^{N_j}), j = 1, \dots, m$ and*

$$v(x) - \varepsilon \leq \sup_{1 \leq j \leq m} (1/N_j) \log \|s_j(x)\|_{N_j \varphi} \leq v(x) \text{ for all } x \in K$$

, where the norm of the section s_j is computed as above.

Proof. (of the Approximation Lemma): Let φ_i be local potentials for the Kähler form ω and let $h = \{h_i = v + \varphi_i\}$ be the positive metric corresponding to v . The inequality in the statement of the lemma is equivalent to

$$h_i - \varepsilon \leq \sup_{1 \leq j \leq m} (1/N_j) \log |s_j(x)| \leq |h_i(x)|, \quad x \in K \cap U_i, \quad i = 1, \dots, l$$

, where $|\cdot|$ is the usual absolute value of a complex number. Let $r \in (0, 1)$ and let χ_r be the function in the class \mathcal{H}^+ on L' associated with the metric $r \cdot h$. For every r the set $\Omega_r = \{\chi_r < 1\}$ is a strictly pseudoconvex neighborhood of the zero section in L' (cf. [GF], VI.6.1). Fix a point $x_0 \in K$ and $\zeta_0 = \theta_i^{-1}(x_0, 1)$. Then $|t| < \chi_r(\zeta_0)$ if and only if $(x_0, t) \in \Omega := \Omega_r$. The function $f(t) = \sum_{n=1}^{\infty} (\chi_r(\zeta_0))^n t^n$, $|t| < 1/\chi_r(\zeta_0)$, $f(0) = 0$ is holomorphic on the analytic set $(\Omega \cap L'_{x_0}) \cup X$ and is identically 0 on X . Let us consider the Remmert reduction of Ω (see [G], Satz 1, or [P], Theorem 2.7 and preceding discussion). That is, we have a Stein space Y and a proper surjective holomorphic map $\Phi : \Omega \rightarrow Y$ with the following properties: (i) Φ has connected fibers; (ii) $\Phi_*(\mathcal{O}_\Omega) = \mathcal{O}_Y$; (iii) the canonical map $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_\Omega(\Omega)$ is an isomorphism; (iv) if $\sigma : \Omega \rightarrow Z$ is a holomorphic map into a Stein space Z then there exists a uniquely determined holomorphic map $\tau : Y \rightarrow Z$ such that $\tau \circ \Phi = \sigma$. The map Φ blows down the zero section of L' . Note that the set $A = \Phi(L'_{x_0} \cup X) = \Phi(L'_{x_0})$ is analytic in Y (by Remmert's Proper Mapping Theorem) and the function $\tilde{f}(\Phi(t)) := f(t)$ is holomorphic on A (by property (ii) of Remmert's reduction). Every analytic set in a complex space is the support of a closed complex subspace (cf. [GR], A.3.5), so we can apply Theorem V.4.4 in

[GR] to conclude that the function \tilde{f} is the restriction to A of a function \tilde{F} that is holomorphic on the Stein space Y . By the properties (ii) and (iii) above, there exists a function F holomorphic on Ω such that $\tilde{F} \circ \Phi = F$. For $t \neq 0$ one can represent F as $F \circ \theta_i^{-1}(x, t) = \sum_{n=1}^{\infty} F_n^{(i)}(x) t^n$, with $F_n^{(i)}$ holomorphic in U_i . We have $F \circ \theta_k^{-1}(x, t) = F \circ \theta_i^{-1}(x, G_{ik}(x)t)$, which gives $F_n^{(i)}(x) = (g_{ik}(x))^n F_n^{(k)}(x)$, i.e., F_n are cocycles corresponding to holomorphic sections of the tensor product L^n over Ω_r . Considering the domain of convergence of the representation for $F \circ \theta_k^{-1}$, $k = 1, \dots, l$, we get $\limsup_{n \rightarrow \infty} |F_n(x)|^{1/n} \leq \exp rh(x)$, $x \in X$. Let $\delta > 0$. By Hartogs's lemma, there exists an $n_\delta > 1$ such that $(1/n) \log |F_n(x)| \leq r \cdot h(x) + \delta$, $x \in K$, $n \geq n_\delta$. For the estimate from below, note that $F_n(x_0) = \chi_r(\zeta_0) = rh(x_0)$ for all n . Since $rh = r(v + \varphi)$ is continuous, there exists an $n_0 \geq n_\delta$ and a neighborhood W_{x_0} of x_0 such that $(1/n_0) \log |F_{n_0}(x)| > rh(x) - \delta$, $x \in W_{x_0}$. Compactness of K and suitable relations between ε, δ and r give holomorphic sections satisfying the conclusion of the lemma. \square

Proof. (of Theorem 3): We mimic the method of proof of Theorem 2.8i in [Bl1]. Let $u \in PSH(X, \omega)$, $u|_K \leq Q$. By Theorem 7.1 in [GZ1], there is a sequence $u_k \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ such that $u_k \searrow u$. Let $\varepsilon > 0$. By Dini's theorem, there exists an integer k_0 such that $u(x) \leq u_k(x) \leq Q(x) + \varepsilon$ for all $x \in K$, $k \geq k_0$. By adding a small multiple of a local Kähler potential in some coordinate neighborhood, we can assume that $dd^c u_k + \omega$ is strictly positive. By the Approximation Lemma, $\exists s_j^{(k)} \in \Gamma(X, L^{N_j^{(k)}})$, $j = 1, \dots, m_k$ such that

$$u_k - 3\varepsilon \leq \sup_{j=1, \dots, m_k} (\log |\exp(-2N_j^{(k)} \varepsilon s_j^{(k)})| / (N_j^{(k)})) \leq (1/n) \log \Phi_n(x),$$

where $n = \max_j N_j^{(k)}$, $j = 1, \dots, m_k$. Hence $u - 4\varepsilon \leq \log \Phi$. The reverse inequality is obvious, since $(1/N) \log \|s\|_{N\varphi}$ defines a positive singular metric on L . \square

Under the assumptions of Theorem 3 we also have the following:

Proposition 8. : Let $\Psi(x) = \lim_{n \rightarrow \infty} \psi_n(x) = \sup_{n \geq 1} \psi_n(x)$, with $\psi_n(x) = \sup\{\|s\|_{n\varphi}(x), s \in \Gamma(X, L^n), \sup_K^\circ(\exp(-nQ)\|s\|_{n\varphi}) \leq 1\}$ and $\sup_K^\circ(f) := \inf\{\sup_{K \setminus P}(f) : P \subset K, P \text{ is } PSH(X, \omega) - \text{polar}\}$. Then

$$V_{K, \omega, Q}^* = (\log \Psi_K)^*.$$

The proof proceeds exactly like that of [Bl1], Theorem 2.8(ii), provided we have the domination principle on a compact Kähler manifold of dimension N (cf. [Kl], cor. 3.7.5 and prop.5.5.1 [BT2], cor.4.5, [Ta], for versions on open subsets of \mathbb{C}^N). In our proof we will assume that one of functions is in $L^\infty(X)$, since this is the case we need. A more general version was recently proved independently as Proposition 2.7 in [BB]. Proofs of the domination principle rely on the comparison principle, which was established in [GZ2](cf. also [Ko], [HKH]) for the class of functions $\mathcal{E}(X, \omega)$ defined therein, which contains $L^\infty(X)$. Recall the following result, which allows us to apply the comparison and domination principles in the weighted theory

Proposition 9. : If K is not $PSH(X, \omega)$ -polar and Q is continuous, then $V_{K, \omega, Q}^* \in PSH(X, \omega) \cap L^\infty(X)$. In particular, the complex Monge-Ampère operator $(\omega_{V_{K, Q}})^n$ is well defined and satisfies $(\omega_{V_{K, Q}})^N = 0$ in $X \setminus \overline{K}$.

Proof. The proof proceeds as that of [GZ1], Theorem 4.2.2, and uses Proposition 2. \square

Now we may state and prove the required domination principle.

Theorem 4. (*Domination Principle*): *Let $u, v \in PSH(X, \omega)$ with $v \in L^\infty(X)$ be such that*

$$\int_{\{u < v\}} (\omega + dd^c u)^N = 0.$$

Then $u \geq v$ in X .

Proof. The following argument was communicated to us by Ahmed Zeriahi as a replacement for an earlier incorrect proof. It is enough to prove that $u \geq v$ on a set of full ω -volume in X . We can assume that v is negative everywhere on X . Then for all $s, t > 0$, $\{u - v \leq -s - t\} \subset \{u - v \leq -s - tv\}$, which for small t is still a subset of $\{u - v < 0\}$. Then, by Lemma 2.2 in [EGZ],

$$0 = \int_{\{u - v < -s - tv\}} (\omega + dd^c u)^N \geq t^N \text{Cap}\{u - v \leq -s - t\},$$

where Cap is the Monge-Ampere capacity defined in [GZ1] (Definition 2.4). Proposition 2.5(1) in [GZ1] implies that $\text{Vol}\{u - v \leq -s - t\} = 0$ for $s, t > 0$, t small, hence $\text{Vol}\{u - v < 0\} = 0$. \square

Finally, we are interested in how weighted pluricomplex Green functions change under a holomorphic map $f : X \rightarrow X$, where X is a compact Kähler manifold (not necessarily projective algebraic) with a closed real $(1, 1)$ -current ω on X with continuous local potentials (not necessarily a Kähler form). Proposition 4.4.5 in [GZ1] states that if $f : X \rightarrow X$ is holomorphic, and $K \subset X$ is a Borel set, then $V_{f(K), \omega} \circ f \leq V_{K, f^* \omega}$. The proof applies also to the weighted pluricomplex Green function and gives the following:

Proposition 10. *Let X, ω, K be as above and let Q be a mild function on X . Then $V_{f(K), \omega, Q} \circ f \leq V_{K, f^* \omega, Q \circ f}$ in X .*

Below, we establish a relation between the pullback of $V_{K, \omega, Q}$ by a surjective holomorphic map $f : X \rightarrow X$ and $V_{f^{-1}(K), \omega, \tilde{Q}}$ with an appropriate function \tilde{Q} . For a function $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ let us define $f_* u(x) = \sup\{u(y) : y \in f^{-1}(x)\}$. This is a well defined function, since $f^{-1}(x)$ is compact. Also, let $f^* u = u \circ f$. The following theorem generalizes Theorem 2.12 in [Bra] and Theorem 1 in [St1] (it yields both as special cases):

Theorem 5. : *Assume that there exist α and β , $1 < \alpha \leq \beta$, such that*

$$\alpha f_*(PSH(X, \omega)) \subset PSH(X, \omega)$$

and

$$f^*(PSH(X, \omega)) \subset \beta \cdot PSH(X, \omega).$$

Then for every Borel set $K \subset X$ and every mild function Q on X ,

$$\alpha V_{f^{-1}(K), \omega, f^* Q / \alpha}(x) \leq V_{K, \omega, Q} \circ f(x) \leq \beta V_{f^{-1}(K), \omega, f^* Q / \beta}.$$

Proof. Let $u \in PSH(X, \omega)$ be such that $\alpha u \leq f^*Q$ on $f^{-1}(K)$. Then $v = \alpha f_*u$ is in $PSH(X, \omega)$ and satisfies $v \leq Q$ on X . Moreover, $\alpha u(x) \leq v(f(x)) \leq V_{K, \omega, Q}(f(x))$, which gives the first inequality. For the second one, if $u \in PSH(X, \omega)$ satisfies $u \leq Q$ on K , then by assumption $(1/\beta)f^*u$ is in $PSH(X, \omega)$ and $(1/\beta)f^*u \leq (1/\beta)f^*Q$ on $f^{-1}(K)$, which gives the conclusion. \square

On $X = \mathbb{CP}^N$, the assumptions of Theorem 5 are equivalent to assumptions about growth of f made in Theorem 2.12 in [Bra] or its unweighted counterpart, Theorem 5.3.1 in [Kl]. Details may be found in Theorem 1 in [St1]) and its proof. The main theorem in [St2] has conditions equivalent to the assumption $\alpha f_*PSH(X, \omega) \subset PSH(X, \omega)$ when $X \hookrightarrow \mathbb{CP}^N$ is a projective algebraic manifold and ω is the pullback of the Fubini-Study form by the embedding \hookrightarrow . One of the conditions is that f has an attracting divisor in X , so in fact the assumption is quite strong.

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